Spherical functions on the space of p-adic quaternion hermitian matrices (Analytic and Arithmetic Theory of Automorphic Forms)

Hironaka, Yumiko

数理解析研究所講究録 第2100巻 2019年 121-133

Spherical functions on the space of p-adic quaternion hermitian matrices *

Yumiko Hironaka
Department of Mathematics,
Faculty of Education and Integrated Sciences, Waseda University
Nishi-Waseda, Tokyo, 169-8050, JAPAN,

§0 Introduction

Let $\mathbb{G}$ be a reductive linear algebraic group defined over k, and $\mathbb{X}$ be an affine algebraic variety defined over k which is $\mathbb{G}$-homogeneous, where and henceforth k stands for a non-archimedian local field of characteristic 0. The Hecke algebra $\mathcal{H}(G, K)$ of G with respect to K acts by convolution product on the space of $\mathcal{C}^\infty(K\backslash X)$ of K-invariant $\mathbb{C}$-valued functions on X, where K is a maximal compact open subgroup of $G = \mathbb{G}(k)$ and $X = \mathbb{X}(k)$. A nonzero function in $\mathcal{C}^\infty(K\backslash X)$ is called *a spherical function on* X if it is a common $\mathcal{H}(G, K)$-eigen function.

Sherical functions on the spaces of susquilinear forms are particularly interenting, sincce they have a close relation to classical number theory, e.g., local densities of representations of corresponding forms. For the case of alternating forms and unramified hermitian forms, the main terms of the explicit formulas are related to Hall-Littlewood polynomials of type A_n, which are well studied. Hence it is possible to extract local densities of forms. For the case of unitary hermitian forms, the main terms of the explicit formulas are related to Hall-Littlewood polynomials of type C_n.

In the present paper, we consider the space X of quaternion hermitian forms on a $\mathfrak{p}$-adic field k of odd residual characteristic, define typical spherical functions and describe the relation to the local densities of forms in §1 and §2. Then we study the functional equations and location of possible poles and zeros of the spherical functions in §3, and give explicit formulas by a general method introduced in [H4] in §4. In this case we obtain a different kind of symmetric polynomials as the main terms of the spherical functions. In §5, we put some remarks and recall previous results on sesquilinear forms.

*This research is partially supported by Grant-in-Aid for Scientific Research (C) JP16K05081. A full version of this paper will appear in elsewhere.

§1 The space X and spherical functions on it

Let k be a $\mathfrak{p}$-adic field of odd residual characteristic, and denote by $\mathfrak{o}$ the ring of integers, π a fixed prime element, and q the cardinality of $\mathfrak{o}/(\pi)$. Let D be a division quaternion algebra over k and $\mathcal{O}$ be the maximal order in D. Then there is an unramified quadratic extension k' of k in D, for which $k' = k(\epsilon)$, $\epsilon^2 \in \mathfrak{o}^\times$ and we may take the prime element Π of D such that $\Pi^2 = \pi$, $\Pi\epsilon = -\epsilon\Pi$ and the set $\{1, \epsilon, \Pi, \Pi\epsilon\}$ forms a standard basis of $\mathcal{O}/\mathfrak{o}$. Then the standard involution $*$ on D is defined by

$$\alpha = a + b\epsilon + c\Pi + d\Pi\epsilon \longmapsto \alpha^* = a - b\epsilon - c\Pi - d\Pi\epsilon, \quad (a, b, c, d \in k), \tag{1.1}$$

and where $\alpha\alpha^* \in k$.

There is a k-algebra inclusion $\varphi : D \longrightarrow M_2(k')$ such that

$$\begin{aligned}
&\alpha(1,\ \Pi) = (1,\ \Pi)\varphi(\alpha), \varphi(\alpha) = \begin{pmatrix} a+b\epsilon & (c-d\epsilon)\pi \\ c+d\epsilon & a-b\epsilon \end{pmatrix} \in M_2(k'),\\
&\det(\varphi(\alpha)) = \alpha\alpha^* = \mathrm{N}_{\mathrm{rd}}(\alpha) \in k,\\
&\mathrm{trace}(\varphi(\alpha)) = \alpha + \alpha^* = \mathrm{T}_{\mathrm{rd}}(\alpha) \in k,
\end{aligned} \tag{1.2}$$

where α is written as in (1.1), N_{rd} is the reduced norm, and T_{rd} is the reduced trace. Based on φ, we have a k-algebra inclusion $\varphi_n : M_n(D) \longrightarrow M_{2n}(k')$ and the reduced norm and trace of an element of $A \in M_n(D)$ are give by

$$\mathrm{N}_{\mathrm{rd}}(A) = \det(\varphi_n(A)),\ \mathrm{T}_{\mathrm{rd}}(A) = \mathrm{trace}(\varphi_n(A))\ (\in k). \tag{1.3}$$

In particular, we see

$$\mathrm{N}_{\mathrm{rd}}(a) = \det(a)^2,\ \mathrm{T}_{\mathrm{rd}}(a) = 2\mathrm{trace}(a), \quad \text{for} \quad a \in M_n(k). \tag{1.4}$$

Since N_{rd} and T_{rd} do not depend on the choice of splitting fields, we will use also another k-algebra inclusion $\varphi_n' : M_n(D) \longrightarrow M_{2n}(k(\Pi))$ based on

$$\begin{aligned}
&\alpha(1,\ \epsilon) = (1,\ \epsilon)\varphi'(\alpha), \quad \varphi'(\alpha) = \begin{pmatrix} a+c\Pi & (b+d\Pi)\epsilon^2 \\ b-d\Pi & a-c\Pi \end{pmatrix} \in M_2(k(\Pi)),\\
&\mathrm{N}_{\mathrm{rd}}(A) = \det(\varphi_n'(A)),\ \mathrm{T}_{\mathrm{rd}}(A) = \mathrm{trace}(\varphi_n'(A))\ (\in k).
\end{aligned} \tag{1.5}$$

We extend the involution $*$ on $A = (a_{ij}) \in M_{mn}(D)$ by $A^* = (a_{ji}^*) \in M_{nm}(D)$. We define the space X_n of quaternion hermitian forms and the action of $G_n = GL_n(D)$ as follows

$$\begin{aligned}
&X_n = \{x \in G_n \mid x^* = x\},\\
&g \cdot x = gxg^* = x[g^*], \quad \text{for } (g, x) \in G_n \times X_n.
\end{aligned} \tag{1.6}$$

Denote by K_n the maximal order in G_n, i.e., $K_n = G_n(\mathcal{O})$. Then, it is known ([Jac]) the set $K_n\backslash X_n$ of K_n-orbits in X_n is bijectively correspond to

$$\Lambda_n = \left\{ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \;\middle|\; \begin{array}{l} \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n, \\ \text{if } \alpha_i \text{ is odd, then } \sharp\{j \mid \alpha_j = \alpha_i\} \text{ is even} \end{array} \right\}. \tag{1.7}$$

In fact, we associate each $\alpha \in \Lambda_n$ with the matrix $\pi^\alpha \in X_n$ as follows. Writing

$$\alpha = (\underbrace{\alpha_1, \dots, \alpha_1}_{\ell_1}, \dots, \underbrace{\alpha_r, \dots, \alpha_r}_{\ell_r}), \quad \ell_i > 0, \ \sum_i \ell_i = n, \tag{1.8}$$

we set

$$\begin{aligned} \pi^\alpha &= \langle \pi^{\alpha_1^{\ell_1}} \rangle \perp \cdots \perp \langle \pi^{\alpha_t^{\ell_r}} \rangle, \quad \langle \pi^{\alpha_i^{\ell_i}} \rangle \in X_{\ell_i}, \\ \langle \pi^{\alpha_i^{\ell_i}} \rangle &= \begin{cases} Diag(\pi^e, \dots, \pi^e) & \text{if } \alpha_i = 2e, \\ \begin{pmatrix} 0 & \pi^e \Pi \\ -\pi^e \Pi & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & \pi^e \Pi \\ -\pi^e \Pi & 0 \end{pmatrix} & \text{if } \alpha_i = 2e+1. \end{cases} \end{aligned} \tag{1.9}$$

For $g \in G_n$, we denote by $g^{(i)}$ the upper left $i \times i$-block of g, $\ 1 \le i \le n$. For $x \in X_n$, $x^{(i)} = x^{(i)*}$ and $x^{(i)} \in X_i$ if $\mathrm{N}_{\mathrm{rd}}(x^{(i)}) \neq 0$. Because of the K_n-orbit decomposition of X_n, we see

$$\begin{aligned} &\mathrm{N}_{\mathrm{rd}}(\pi^\alpha) = \pi^{|\alpha|}, \quad (|\alpha| = \sum_{i=1}^n \alpha_i \in 2\mathbb{Z}), \\ &\mathrm{N}_{\mathrm{rd}}(x) \in k^2, \quad (x \in X_n). \end{aligned} \tag{1.10}$$

We set B_n the Borel subgroup of G_n consisting of lower triangular matrices. Since $(p \cdot x)^{(i)}) = p^{(i)} \cdot x^{(i)}$, we see for $(p, x) \in B_n \times X_n$ and i

$$\mathrm{N}_{\mathrm{rd}}((p \cdot x)^{(i)}) = \psi_i(p)^2 \mathrm{N}_{\mathrm{rd}}(x^{(i)}), \quad \psi_i(p) = \mathrm{N}_{\mathrm{rd}}(p^{(i)}). \tag{1.11}$$

For $x \in X_n$ and each i with $1 \le i \le n$, set

$$d_i(x) \in k, \quad \text{by } \ d_i(x)^2 = \mathrm{N}_{\mathrm{rd}}(x^{(i)}), \ 1 \le i \le n, \tag{1.12}$$

then each $d_i(x)$ is a B_n-relative invariant associated with k-rational character ψ_i, $1 \le i \le n$. We define spherical function $\omega(x; s)$, for $x \in X_n$ and $s \in \mathbb{C}^n$, set

$$\omega(x; s) = \int_{K_n} |\mathbf{d}(k \cdot x)|^s \, dk, \quad |\mathbf{d}(y)|^s = \begin{cases} \prod_{i=1}^n |d_i(y)|^{s_i} & \text{if } \ y \in X_n^{op} \\ 0 & \text{otherwise}, \end{cases} \tag{1.13}$$

where dk is the normalized Haar measure on K_n, $|\ |$ is the absolute value on k, and

$$X_n^{op} = \{ x \in X_n \mid d_i(x) \neq 0, \ 1 \le i \le n \}. \tag{1.14}$$

The integral in (1.13) is absolutely convergent if $\mathrm{Re}(s_i) \ge 0$, $1 \le i \le n-1$, and continued to a rational function of $q^{s_1}, \dots, q^{s_n}$. It is easy to see that $\omega(x; s)$ is K_n-invariant and becomes a common eigenfunction with respect to the Hecke algebra $\mathcal{H}(G, K)$, in fact

$$\begin{aligned} (f * \omega(\ ; s))(x) \ \ &\left(= \int_{G_n} f(g) \omega(g^{-1} \cdot x; s) dg \right) \\ &= \lambda_s(f) \omega(x; s), \quad (f \in \mathcal{H}(G, K)). \end{aligned} \tag{1.15}$$

Here

$$\begin{aligned} \lambda_s : \ \mathcal{H}(G,K) &\longrightarrow \mathbb{C}(q^{s_1},\dots,q^{s_n}), \\ f &\longmapsto \int_{B_n} f(p) \prod_{i=1}^{n} |\psi_i(p)|^{-s_i}\, \delta(p) dp, \end{aligned} \tag{1.16}$$

where dp is the left invariant measure on B_n with modulus character δ. The Weyl group S_n of G_n acts on $\{s_1,\dots,s_n\}$ through its action on the rational characters $\{\,|\psi_i|^{s_i} \mid 1 \le i \le n\}$. It is convenient to introduce a new variable $z \in \mathbb{C}^n$ related to $s \in \mathbb{C}^n$ by

$$s_i = -z_i + z_{i+1} - 2 \ (1 \le i \le n-1), \quad s_n = -z_n + n - 1, \tag{1.17}$$

and denote $\omega(x;s) = \omega(x;z)$ and $\lambda_s = \lambda_z$. Then S_n acts on $\{z_1,\dots,z_n\}$ by permutation, and the $\mathbb{C}$-algebra map λ_z is the Satake isomorphism

$$\lambda_z : \mathcal{H}(G,K) \xrightarrow{\sim} \mathbb{C}[q^{\pm z_1},\dots,q^{\pm z_n}]^{S_n}. \tag{1.18}$$

§2 Local densities and spherical functions

We state the induction theorem (Theorem 2.1) of spherical functions, with which we may regard spherical functions as generating functions of local densities of representations. We start with the definition of local densities. For $A \in X_m$ and $B \in X_n$ with $m \ge n$, we define

$$\mu(B,A) = \lim_{\ell\to\infty} \frac{N_\ell(B,A)}{q^{\ell n(4m-2n+1)+n(n-1)}}, \tag{2.1}$$

$$\mu^{pr}(B,A) = \lim_{\ell\to\infty} \frac{N_\ell^{pr}\ell(B,A)}{q^{\ell n(4m-2n+1)+n(n-1)}}, \tag{2.2}$$

where

$$\begin{aligned} &N_\ell(B,A) = \sharp\left\{u \in M_{mn}(\mathcal{O}/\mathcal{P}^{2\ell}) \mid A[u] - B \in H_n(\mathcal{P},\ell)\right\}, \\ &N_\ell^{pr}(B,A) = \sharp\left\{u \in M_{mn}^{pr}(\mathcal{O}/\mathcal{P}^{2\ell}) \mid A[u] - B \in H_n(\mathcal{P},\ell)\right\}, \\ &H_n(\mathcal{P},\ell) = \left\{A = (a_{ij}) \in M_n(\mathcal{O}) \mid A = A^*,\ a_{ii} \in \mathfrak{p}^\ell, a_{ij} \in \mathcal{P}^{2\ell-1},\ ({}^\forall i,j)\right\}, \\ &M_{mn}^{pr}(\mathcal{O}/\mathcal{P}^{2\ell}) = GL_m(\mathcal{O}/\mathcal{P}^{2\ell}) \begin{pmatrix} 1_n \\ 0 \end{pmatrix}, \qquad \mathcal{P} = \Pi\mathcal{O},\ \mathfrak{p} = \pi\mathfrak{o}. \end{aligned}$$

Set $\Lambda_n^+ = \{\alpha \in \Lambda_n \mid \alpha_n \ge 0\}$. Then

$$X_n(\mathcal{O})\big(= X_n \cap M_n(O)\big) = \cup_{\alpha\in\Lambda_n^+} K_n \cdot \pi^\alpha.$$

For $r \in \mathbb{Z}$, $x \in X_n$ and $y \in X_m$ with $m \geq n$, we see

$$\mu^{(pr)}(\pi^r x, \pi^r y) = q^{rn(2n-1)}\mu^{(pr)}(x, y), \tag{2.3}$$

$$\omega(\pi^r x; s) = q^{-\sum_{i=1}^n is_i}\omega(x; s) = q^{r(z_1+\cdots+z_n)}\omega(x; s), \tag{2.4}$$

where $\mu^{(pr)}(\ ,\ )$ means that the identity holds both local density $\mu(\ ,\ )$ and primitive local density $\mu^{pr}(\ ,\ )$.

Theorem 2.1 *Let $m > n$. Then, for any $\xi \in X_m^+$, one has*

$$\begin{aligned} &\omega(\pi^\xi; s_1, \ldots, s_n, 0, \ldots, 0) \\ &= \frac{w_n(q^{-2})w_{m-n}(q^{-2})}{w_m(q^{-2})} \times \sum_{\alpha \in \Lambda_n^+} \frac{\mu^{pr}(\pi^\alpha, \pi^\xi)}{\mu(\pi^\alpha, \pi^\alpha)} \cdot \omega(\pi^\alpha; s_1 \ldots, s_n) \\ &= \frac{w_n(q^{-2})w_{m-n}(q^{-2})}{w_m(q^{-2})} \prod_{i=1}^n (1 - q^{-(s_i+\cdots+s_n+2m-2i+2)}) \times \sum_{\alpha \in \Lambda_n^+} \frac{\mu(\pi^\alpha, \pi^\xi)}{\mu(\pi^\alpha, \pi^\alpha)} \cdot \omega(\pi^\alpha; s_1 \ldots, s_n), \end{aligned}$$

where $w_m(t) = \prod_{i=1}^m (1 - t^i)$.

The above theorem can be proved in a similar way to the case for alternating, hermitian or symmetric forms (cf. [HS1], [H1]). For the present case the result is proved in the master thesis of Y. Ohtaka ([Oh]) in a slightly different definition, and he used it to obtain the explicit formula of spherical functions of size 2.

In general, it is not easy to obtain the value of (primitive) local density in a good form. The following formula by using character sum is useful for the calculation. For $B = (b_{ij}), C = (c_{ij}) \in X_n$, set

$$\langle B, C \rangle = \sum_{i=1}^n b_{ii}c_{ii} + \sum_{1 \leq i < j \leq n} \mathrm{T_{rd}}(b_{ij}c_{ij}) \in k. \tag{2.5}$$

Proposition 2.2 *Let $\ell \geq 1$ and take a character $\chi = \chi_\ell$ of $\mathfrak{o}/\mathfrak{p}^\ell$ such that χ is nontrivial on $\mathfrak{p}^{\ell-1}/\mathfrak{p}^\ell$. For $A \in X_m^+$ and $B \in X_n^+$ with $m \geq n$, one has*

$$N_\ell^{(pr)}(B, A) = q^{-\ell n(2n-1)} \sum_{\substack{Y \in M_n(\mathcal{O}/\mathcal{P}^{2\ell}) \\ Y = Y^*}} \sum_{X \in M_{mn}^{(pr)}(\mathcal{O}/\mathcal{P}^{2\ell})} \chi(\langle A[X] - B, Y \rangle). \tag{2.6}$$

It is not so difficult to obtain the density of itself $\mu(\pi^\alpha, \pi^\alpha) = \mu^{pr}(\pi^\alpha, \pi^\alpha)$, and we have the following result.

Proposition 2.3 *Assume $\alpha \in \Lambda_n$ is given as in (1.8). Then one has*

$$\mu(\pi^\alpha, \pi^\alpha) = q^{2n(\alpha)+\frac{1}{2}|\alpha|+\frac{1}{2}\sum_{i:2\nmid\alpha_i}\ell_i} \prod_{i=1}^r \left\{ \begin{array}{ll} w_{\ell_i}(-q^{-1}) & \text{if } 2 \mid \alpha_i \\ w_{\frac{\ell_i}{2}}(q^{-4}) & \text{if } 2 \nmid \alpha_i \end{array} \right\}, \tag{2.7}$$

where

$$w_\ell(t) = \prod_{i=1}^\ell (1 - t^i). \tag{2.8}$$

We define an integral transform F_0 on the Schwartz space

$$\mathcal{S}(K\backslash X) = \{\varphi : X \longrightarrow \mathbb{C} \mid \text{left } K\text{-invariant, compactly supported}\},$$

by using spherical function $\omega(x; z)$ as the kernel function. We will modify F_0 into F in §3.

Proposition 2.4 *For each $\varphi \in \mathcal{S}(K\backslash X)$, set*

$$\begin{array}{rccl} F_0 : & \mathcal{S}(K\backslash X) & \longrightarrow & \mathbb{C}(q^{s_1}, \ldots, q^{s_n}), \\ & \varphi & \longmapsto & \int_X \varphi(x)\omega(x; s)dx, \end{array} \tag{2.9}$$

where dx is a G-invariant measure on X. Then the spherical Fourier transform F_0 is injective and compatible with the action of $\mathcal{H}(G, K)$:

$$F_0(f * \varphi) = \lambda_z(f)F_0(\varphi), \quad f \in \mathcal{H}(G, K),\ \varphi \in \mathcal{S}(K\backslash X),$$

where λ_z is defined in (1.18).

The injectivity of F_0 is proved by using the lemma below and induction on the size n. The similar lemma for symmetric forms and hermitian forms was used in [H1] to prove the injectivity, and the original lemma for symmetric forms had proved by Kitaoka ([Ki]). We define an order $\geq$ in Λ_n by

$$\gamma \geq \alpha \Longleftrightarrow \gamma = \alpha \text{ or } \gamma_{n-i} = \alpha_{n-i},\ 1 \leq i < r, \text{ and } \gamma_{n-r} > \alpha_{n-r} \text{ for some } r \geq 0.$$

Lemma 2.5 *Let n be an integer with $n \geq 2$. For any $\alpha \in \Lambda_n^+$, there exists $\beta \in \Lambda_{n-1}^+$ which satisfies the following properties.*

(1) $\mu^{pr}(\pi^\beta, \pi^\alpha) \neq 0$.

(2) *If $\gamma \in \Lambda_n^+$ satisfies*

$$\text{(i) } |\gamma| = |\alpha|, \quad \text{(ii) } \gamma \geq \alpha \quad \textit{and} \quad \text{(iii) } \mu^{pr}(\pi^\beta, \pi^\gamma) \neq 0,$$

then $\gamma = \alpha$.

§3 Functional equations of spherical functions

First we note the result for size 2, which can be obtained by Theorem 2.1.

Proposition 3.1 *For any $\alpha \in \Lambda_2$, one has*

$$\omega(\pi^\alpha; z) = \begin{cases} \dfrac{q^{\langle \lambda,\, z_0 \rangle}}{1+q^{-2}} \cdot \dfrac{1}{q^{z_2} - q^{z_1+1}} \displaystyle\sum_{\sigma \in S_2} \sigma \left(q^{\langle \lambda,\, z \rangle} \frac{(q^{z_1} - q^{z_2-2})(q^{z_1} - q^{z_2+1})}{q^{z_1} - q^{z_2}} \right) & \text{if } \alpha = 2\lambda, \\ q(1-q^{-1}) \dfrac{q^{e(z_1+z_2)}}{q^{z_2} - q^{z_1+1}} & \text{if } \alpha = (2e-1, 2e-1), \end{cases}$$

where $z_0 = (1, -1) \leftrightarrow s = \mathbf{0}$, $\langle \lambda,\, z \rangle = \lambda_1 z_1 + \lambda_2 z_2$ and S_2 acts on $\{z_1, z_2\}$ by permutation. Especially, for any $x \in X_2$, one has

$$(q^{z_2} - q^{z_1+1}) \cdot \omega(x; z) \in \mathbb{C}[q^{\pm z_1}, q^{\pm z_2}]^{S_2}. \tag{3.1}$$

We use the similar method for the study the functional equations and holomorphy for general n to the case of unramified hermitian forms. We introduce, for $\varphi \in \mathcal{S}(K \backslash X)$

$$\Phi(s, \varphi) = \int_X |\mathbf{d}(x)|^s \varphi(x) dx, \quad |\mathbf{d}(x)|^s = \begin{cases} \prod_{i=1}^n |d_i(x)|^{s_i} & \text{if } x \in X^{op} \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

where dx is a G-invariant measure on X. The integral is absolutely convergent if $\mathrm{Re}(s_i) \geq 0$, $1 \leq i \leq n-1$, and continued to a rational function of $q^{s_1}, \ldots, q^{s_n}$. Keeping the relation (1.17) between s and z, we denote $\Phi(z, \varphi)$.

Lemma 3.2 *Let $n \geq 2$ and take α with $1 \leq \alpha \leq n-1$. Then for any $\varphi \in \mathcal{S}(K \backslash X)$, one has*

$$\Phi(z, \varphi) = \int_{X^{op}} \prod_{i \neq \alpha, \alpha+1} |d_i(x)|^{s_i} \cdot \prod_{j = \alpha \pm 1} |d_j(x)|^{\frac{s_\alpha}{2} + s_j} \cdot \varphi(x) \cdot \omega^{(2)}(\widetilde{x}; s_\alpha, -\frac{s_\alpha}{2}) dx, \tag{3.3}$$

where $\widetilde{x}$ to be the lower right (2×2)-block of $(x^{(\alpha+1)})^{-1}$ and $\omega^{(2)}(y; s)$ indicates the spherical function of size of 2.

Proposition 3.3 *The function*

$$\prod_{1 \leq i < j \leq n} (q^{z_j} - q^{z_i+1}) \times \Phi(z, \varphi)$$

is holomorphic in $\mathbb{C}^n$ and S_n-invariant, hence it is an element of

$$\mathbb{C}[q^{\pm z_1}, \ldots, q^{\pm z_n}]^{S_n}.$$

When we take the characteristic function of $K \cdot x$ for $x \in X_n$ as φ, we have

Theorem 3.4 *$G_n(z) \cdot \omega(x;z)$ is holomorphic for $s \in \mathbb{C}^n$ and S_n-invariant, where*

$$G_n(z) = \prod_{1 \le i < j \le n} (q^{z_j} - q^{z_i+1}).$$

Especially one has

$$G_n(z) \cdot \omega(x;z) \in \mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^{S_n}.$$

By Theorem 3.4, we modify the spherical Fourier transform F_0 in (2.9) as follows.

Corollary 3.5 *Define the normalized spherical Fourier transform by*

$$\begin{array}{rccl} F: & \mathcal{S}(K\backslash X) & \longrightarrow & \mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^{S_n} (= \mathcal{R}, \textit{say}) \\ & \varphi & \longmapsto & \widehat{\varphi}(z) = \displaystyle\int_X \varphi(x) \cdot \omega(x;z) G_n(z) dx. \end{array} \tag{3.4}$$

Then F is an injective $\mathcal{H}(G,K)$-module map, hence one has the commutative diagram

$$\begin{array}{ccccc} \mathcal{H}(G,K) & \times & \mathcal{S}(K\backslash X) & \xrightarrow{*} & \mathcal{S}(K\backslash X) \\ {\scriptstyle \lambda_z}\downarrow\wr & & {\scriptstyle F}\downarrow & \circlearrowleft & {\scriptstyle F}\downarrow \\ \mathcal{R} & \times & \mathcal{R} & \longrightarrow & \mathcal{R}, \end{array} \tag{3.5}$$

where the upper $*$ is the action of $\mathcal{H}(G,K)$ on $\mathcal{S}(K\backslash X)$, the lower arrow is the multiplication in $\mathcal{R}$ and λ_z is the Satake isomorphism defined in (1.18).

§4 Explicit formula for $\omega(x;z)$

As for the explicit formula of $\omega(x;z)$, it suffices to determine at a representative for every K-orbit in X, hence at π^α, $\alpha \in \Lambda_n$ (cf. (1.9)). Since we have obtained the functional equation of $\omega(x;z)$ in a good shape, we may apply the general expression formula given in [H4] of spherical function on homogeneous spaces. We note here that X_n is a single B_n-orbit.

Proposition 4.1 *For general $x \in X$ and $z \in \mathbb{C}^n$, one has*

$$\omega(x;z) = \frac{1}{Q_n \cdot G_n(z)} \times \sum_{\sigma \in S_n} \sigma\left(\gamma_n(z) G_n(z) \delta(x;z)\right). \tag{4.1}$$

Here $G_n(z)$ is given in Theorem 3.4,

$$Q_n = \frac{\prod_{i=1}^n (1-q^{-2i})}{(1-q^{-2})^n},$$

$$\gamma_n(z) = \prod_{1\le i<j\le n} \frac{1-q^{z_i-z_j-2}}{1-q^{z_i-z_j}} = \prod_{i<j} \frac{q^{z_j}-q^{z_i-2}}{q^{z_j}-q^{z_i}},$$

$$\delta(x;z) = \delta(x;s) = \int_U |\mathbf{d}(\nu\cdot x)|^s \, d\nu = \int_{U_1} |\mathbf{d}(\nu\cdot x)|^s \, d\nu,$$

where U is the Iwahori subgroup of K_n associated with the Borel groups B_n.

We note here that $Q_n = \sum_{\sigma\in S_n} [U\sigma U : U]^{-1}$ and $\gamma_n(z)$ are determined by the group $G_n = GL_n(D)$, hence the problem is reduced to the calculation of $\delta(x;z)$. For each $\alpha = (\alpha_i) \in \Lambda_n$, we set

$$\lambda_\alpha = (\lambda_i) \in \Lambda_n \text{ by } \lambda_i = \begin{cases} \dfrac{\alpha_i}{2} & \text{if } 2 \mid \alpha_i \\ \dfrac{\alpha_i+1}{2} & \text{if } 2 \not| \alpha_i \end{cases} \tag{4.2}$$

If α has an odd entry, odd entries appear in pairs. We assume they are

$$\alpha_{\ell_1}, \alpha_{\ell_1+1}, \dots, \alpha_{\ell_k}, \alpha_{\ell_k+1}, \quad \ell_1 < \ell_2 < \cdots < \ell_k, \tag{4.3}$$

and set

$$I_{odd}(\alpha) = \{\ell_1, \dots, \ell_k\}, \quad c_{odd}(\alpha) = (1-q^{-1})^k \cdot q^{\sum_{\ell\in I_{odd}(\alpha)} (n-2\ell+1)}. \tag{4.4}$$

If α has no odd entry we say α is *even*, and set $I_{odd}(\alpha) = \emptyset$ and $c_{odd}(\alpha) = 1$ for convenience. Only if α is even, π^α is diagonal and $\lambda_\alpha = \dfrac{\alpha}{2}$.

We introduce some more notation. Take $j = j_n$ to be an element in K whose anti-diagonal entries are 1 and the others are 0, consider $j\cdot\pi^\alpha \in K\cdot\pi^\alpha \subset X$, and set $jz = (z_n, \dots, z_1)$. We write

$$z_0 = (-n+1, -n+3, \dots, n-1) \in \mathbb{C}^n \tag{4.5}$$

the corresponding value in z-variable to $s = \mathbf{0} \in \mathbb{C}^n$. For $\lambda \in \mathbb{Z}^n$ and $z \in \mathbb{C}^n$, set $\langle\lambda, z\rangle = \sum_{i=1}^n \lambda_i z_i$.

Lemma 4.2 *For any $\alpha \in \Lambda_n$, one has*

$$\delta(j\cdot\pi^\alpha; z) = \frac{c_{odd}(\alpha)\cdot q^{\langle\lambda_\alpha, z_0\rangle + \langle\lambda_\alpha, jz\rangle}}{\prod_{\ell\in I_{odd}(\alpha)} \left(q^{z_{n-\ell+1}} - q^{z_{n-\ell}+1}\right)}. \tag{4.6}$$

The calculation of the above lemma for odd α is rather troublesome. By proposition 4.1 and Lemma 4.2, we obtain the following explicit formulas of spherical functions.

Theorem 4.3 *For any $\alpha \in \Lambda_n$, one has*

$$\omega(\pi^\alpha; z) = \frac{c_{odd}(\alpha) \cdot q^{\langle \lambda_\alpha, z_0 \rangle}}{Q_n \cdot G_n(z)} \times \sum_{\sigma \in S_n} \sigma \left(\frac{q^{\langle \lambda_\alpha, z \rangle}}{\prod_{\ell \in I_{odd}(\alpha)} (q^{z_\ell} - q^{z_{\ell+1}+1})} \prod_{i<j} \frac{(q^{z_i} - q^{z_j+1})(q^{z_i} - q^{z_j-2})}{q^{z_i} - q^{z_j}} \right). \tag{4.7}$$

§5 Remarks

- We take the main term of spherical function for each $\alpha \in \Lambda_n$, and set

$$\Psi_\alpha(z) = \sum_{\sigma \in S_n} \sigma \left(\frac{q^{\langle \lambda_\alpha, z \rangle}}{\prod\limits_{\ell \in I_{odd}(\alpha)} (q^{z_\ell} - q^{z_{\ell+1}+1})} \prod_{i<j} \frac{(q^{z_i} - q^{z_j+1})(q^{z_i} - q^{z_j-2})}{q^{z_i} - q^{z_j}} \right). \tag{5.1}$$

  Then we see $\Psi_\alpha(z)$ is holomorphic for $z \in \mathbb{C}^n$ and linearly independent with respect to $\alpha \in \Lambda_n$ (cf. Theorem 3.4, Corollary 3.5).

- In general, the image $\mathrm{Im}(F)$ of the spherical Fourier transform F defined in (3.4) is an ideal in

$$\mathcal{R}_n = \mathbb{C}[q^{\pm z_1}, \ldots, q^{\pm z_n}]^{S_n} \tag{5.2}$$

  generated by $\{\, \Psi_\alpha(z) \mid \alpha \in \Lambda_n \}$ (cf. the commutative diagram (3.5)). For size 2, F is surjective, since $\Psi_{(-1,-1)}(z)$ is constant, and we see F gives an $\mathcal{H}(G, K)$-module isomorphism between $\mathcal{S}(K \backslash X)$ and $\mathcal{R}_2 = \mathbb{C}[q^{z_1} + q^{z_2}, q^{\pm(z_1+z_2)}]$, and we may construct the Plancherel formula.

  In the following, we note some known cases of sesquilinear forms.

- (**The case of alternating forms**, cf. [HS1]): Set $X_n = \{ x \in GL_{2n}(k) \mid {}^t x = -x \}$, $G = GL_{2n}(k)$ and $K = GL_{2n}(\mathcal{O}_k)$. ($k$ admits even characteristic.) Then $K \backslash X_n$ is parametrized by the set

$$\widetilde{\Lambda}_n = \{\, \lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n \} \ (\supset \Lambda_n), \tag{5.3}$$

  and we have known the explicit formula of spherical functions $\omega(x; z)$ on X_n, the main term of $\omega(\pi^\lambda; z)$, where $\pi^\lambda \in X_n$ is associated with $\lambda \in \widetilde{\Lambda}_n$, is given as

$$\Psi_\lambda^{(A)}(z) = \sum_{\sigma \in S_n} \sigma \left(q^{\langle \lambda, z \rangle} \prod_{i<j} \frac{q^{z_i} - q^{z_j-2}}{q^{z_i} - q^{z_j}} \right). \tag{5.4}$$

Then $\Psi_\lambda^{(A)}(z)$ are (constant multiple of specialized) Hall-Littlewood polynomial of type A_n, and it becomes a constant when $\lambda = \mathbf{0}$. Then the normalized spherical Fourier transform F is isomorphic onto $\mathcal{R}_n$, and we have the commutative diagram

$$\begin{array}{ccccc} \mathcal{H}(G,K) & \times & \mathcal{S}(K\backslash X) & \xrightarrow{*} & \mathcal{S}(K\backslash X) \\ \downarrow{\scriptstyle \lambda_z} & & {\scriptstyle F}\downarrow\wr & \circlearrowleft & {\scriptstyle F}\downarrow\wr \\ \mathcal{R}_n & \times & \mathcal{R}_n & \longrightarrow & \mathcal{R}_n, \end{array} \tag{5.5}$$

where the adjusted Satake transform λ_z is surjective and decomposed as $\mathcal{H}(G,K) \xrightarrow{\sim} \mathcal{R}_{2n} \longrightarrow \mathcal{R}_n$. It is known $\mathcal{S}(K\backslash X) = \mathcal{H}(G,K) * \phi_0$ with the characteristic function ϕ_0 of $K \cdot \pi^{\mathbf{0}}$, $\pi^{\mathbf{0}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in X_n$, and the Plancherel formula for F is known.

- **(The case of unramified hermitian forms**, cf. [H1]): Taking an unramified quadratic extension k'/k, set $X_n = \{x \in GL_n(k') \mid x^* = x\}$, where $*$ means the conjugate tranpose, $G = GL_n(k')$ and $K = GL_n(\mathcal{O}_{k'})$. ($k$ admits even characteristic.) Then $K\backslash X_n$ is parametrized by the same $\widetilde{\Lambda}_n$ as in (5.3), and we have known the explicit formula of spherical functions $\omega(x; z)$ on X_n, the main term of $\omega(\pi^\lambda; z)$, where $\pi^\lambda \in X_n$ is associated with $\lambda \in \widetilde{\Lambda}_n$, is given as

$$\Psi_\lambda^{(H)}(z) = \sum_{\sigma \in S_n} \sigma \left(q^{\langle \lambda, z \rangle} \prod_{i<j} \frac{q^{z_i} + q^{z_j - 1}}{q^{z_i} - q^{z_j}} \right). \tag{5.6}$$

Then $\Psi_\lambda^{(H)}(z)$ are (constant multiple of specialized) Hall-Littlewood polynomials of type A_n, where the specialization is different from the case of alternating forms, and it becomes a constant when $\lambda = \mathbf{0}$. Then the normalized spherical Fourier transform F is isomorphic onto $\mathcal{R}_n$, and we have the commutative diagram

$$\begin{array}{ccccc} \mathcal{H}(G,K) & \times & \mathcal{S}(K\backslash X) & \xrightarrow{*} & \mathcal{S}(K\backslash X) \\ {\scriptstyle \lambda_z}\downarrow\wr & & {\scriptstyle F}\downarrow\wr & \circlearrowleft & {\scriptstyle F}\downarrow\wr \\ \mathcal{R}_{0,n} & \times & \mathcal{R}_n & \longrightarrow & \mathcal{R}_n, \end{array} \tag{5.7}$$

where $\mathcal{R}_{0,n} = \mathbb{C}[q^{\pm 2z_1}, \ldots, q^{\pm 2z_n}]^{S_n}$, and λ_z is the (adjusted) Satake isomorphism. Hence one sees $\mathcal{S}(K\backslash X)$ is a free $\mathcal{H}(G,K)$-module of rank 2^n, and the Plancherel formula for F is known.

- **(Relations with local densities**, cf. [H2], [H3], [HS1], [HS2]): There are many works for Hall-Littlewood polynomials of type A_n (original Hall-Littlewood polynomials), and their relations are well known. Hence one may extract local densities from induction theorem of type Theorem 2.1. In the present case, we don't know well about $\Psi_\alpha(z)$ and general local densities.

- **(The case of unitary hermitian forms**, cf. [HK1], [HK2], [H5]) Taking an unramified quadratic extension k'/k, set $G = U(j_m) = \{g \in GL_m(k') \mid g^* j_m g = j_m\}$

and $K = G(\mathcal{O}_{k'})$, where $*$ means the conjugate tranpose and $j_m \in GL_m(k)$ is the matrix whose anti-diagonal entries are 1 and the others are 0. Set $X_m = \{x \in G \mid x^* = x\}$, $n = \left[\frac{m}{2}\right]$, and $e = v_\pi(2)$ with an prime element π in k. We assume $v_\pi(2) \leq 1$ if m is odd and $m \geq 5$. Then $K\backslash X_m$ is parametrized by the set $\widetilde{\Lambda}_n^+ = \left\{ \lambda \in \widetilde{\Lambda}_n \;\middle|\; \lambda_n \geq -e \right\}$, and we have known the explicit formula $\omega(x; z)$ on X_n, the main term of $\omega(\pi^\lambda; z)$, where $\pi^\lambda \in X_n$ is associated with $\lambda \in \widetilde{\Lambda}_n^+$, is given as

$$\Psi_\lambda^{(U)}(z) = \sum_{\sigma \in W} \sigma\left(q^{\langle \lambda + e,\, z\rangle} c(z; \{t\})\right), \quad c(z; \{t\}) = \prod_{\alpha \in \Sigma^+} \frac{1 - t_\alpha q^{\langle \alpha,\, z\rangle}}{1 - q^{\langle \alpha,\, z\rangle}}. \tag{5.8}$$

Here $W \cong S_n \ltimes (\pm 1)^n$ is the Weyl group of G with respect to the Borel subgroup consisting of all the upper triangular matrices, Σ^+ is the set of positive roots, where the root system of G is of type C_n (resp. BC_n) when $m = 2n$ (resp. $m = 2n + 1$), and $t_\alpha \in \{\pm q^{-1}, q^{-2}\}$ is explicitly given depending on the length of α and the parity of m. We see $\Psi_\alpha^{(U)}$ are (constant multiple of specialized) Hall-Littlewood polynomial of type C_n, where the specialization is depend on the parity of m. By the normalized spherical Fourier transform we have the same shape of commutative diagram with (5.7), with

$$\mathcal{R}_n = \mathbb{C}[q^{\pm z_1}, \ldots, q^{\pm z_n}]^W, \quad \mathcal{R}_{0,n} = \mathbb{C}[q^{\pm 2z_1}, \ldots, q^{\pm 2z_n}]^W. \tag{5.9}$$

Hence one sees $\mathcal{S}(K\backslash X)$ is a free $\mathcal{H}(G, K)$-module of rank 2^n, and the Plancherel formula for F is known.

References

[H1] Y. Hironaka: Spherical function of hermitian and symmetric forms I, *Japan. J. Math.*$\mathbf{14}$(1988), 203 – 223.

[H2] Y. Hironaka: Spherical function and local densities on hermitian forms, *J. Math. Soc. Japan* $\mathbf{51}$(1999), 553 – 581.

[H3] Y. Hironaka: Local zeta functions on hermitian forms and its application to local densities, *J. Number Theory*$\mathbf{71}$(1998), 40 – 64.

[H4] Y. Hironaka: Spherical functions on p-adic homogeneous spaces, "Algebraic and Analytic Aspects of Zeta Functions and L-functions – Lectures at the French-Japanese Winter School (Miura, 2008)– ", *MSJ Memoirs* $\mathbf{21}$(2010), 50 – 72.

[H5] Y. Hironaka: Harmonic analysis on the space of p-adic unitary hermitian matrices, mainly for dyadic case, *Tokyo J. Math.*$\mathbf{40}$(2017), 517 – 564.

[HK1] Y. Hironaka and Y. Komori: Spherical functions on the space of p-adic unitary hermitian matrices, *Int. J. Number Theory* **10**(2014). 513 – 558; Math arXiv:1207.6189

[HK2] Y. Hironaka and Y. Komori: Spherical functions on the space of p-adic unitary hermitian matrices II, the case of odd size, *Commentarii Math. Univ. Sancti Pauli* **63**(2014), 47 – 78; Math arXiv:1403.3748

[HS1] Y. Hironaka and F. Sato: Spherical functions and local densities of alternating forms, *American Journal of Mathematics* **110**(1988), 473 – 512.

[HS2] Y. Hironaka and F. Sato: Local densities of alternating forms, *Journal of Number Theory* **33**(1989), 32 – 52.

[Jac] R. Jacobowitz: Hermitian forms over local fields, *Amer. J. Math.***84**(1962), 12 – 22.

[Ki] Y. Kitaoka: Representations of quadratic forms and their application to Selberg's zeta functions, *Nagoya Math. J.***63**(1976), 153 –162.

[M1] I. G. Macdonald: *Symmetric Functions and Hall Polynpomials*, Oxfor Science Publ., 1979.

[M2] I. G. Macdonald: Orthogonal polynomials associated with root systems, *Séminaire Lothanringien de Combinatoire* **45**(2000). Article B45a.

[Oh] 大高 靖弘：Quaternion hermitian matrices の空間上の球関数 (修士論文), 2004.1.